

# THE RICCATI DIFFERENTIAL EQUATION AND A DIFFUSION-TYPE EQUATION

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**ABSTRACT.** We construct an explicit solution of the Cauchy initial value problem for certain diffusion-type equations with variable coefficients on the entire real line. The corresponding Green function (heat kernel) is given in terms of elementary functions and certain integrals involving a characteristic function, which should be found as an analytic or numerical solution of the second order linear differential equation with time-dependent coefficients. Some special and limiting cases are outlined. Solution of the corresponding non-homogeneous equation is also found.

## 1. INTRODUCTION

In this paper we discuss explicit solution of the Cauchy initial value problem for the one-dimensional heat equation on the entire real line

$$\frac{\partial u}{\partial t} = Q\left(\frac{\partial}{\partial x}, x, t\right)u, \quad (1.1)$$

where the right hand side is a quadratic form  $Q(p, x)$  of the coordinate  $x$  and the operator of differentiation  $p = \partial/\partial x$  with time-dependent coefficients; see equation (2.1) below. The case of a corresponding Schrödinger equation is investigated in [6]. In this approach, several exactly solvable models are classified in terms of elementary solutions of a characterization equation given by (2.13) below. Solution of the corresponding non-homogeneous equation is obtained with the help of the Duhamel principle. These exactly solvable cases may be of interest in a general treatment of the nonlinear evolution equations; see [3], [4], [5], [26] and references therein. Moreover, these explicit solutions can also be useful when testing numerical methods of solving the semilinear heat equations with variable coefficients.

## 2. SOLUTION OF A CAUCHY INITIAL VALUE PROBLEM: SUMMARY OF RESULTS

The fundamental solution of the diffusion-type equation of the form

$$\frac{\partial u}{\partial t} = a(t) \frac{\partial^2 u}{\partial x^2} - b(t) x^2 u + c(t) x \frac{\partial u}{\partial x} + d(t) u + f(t) x u - g(t) \frac{\partial u}{\partial x}, \quad (2.1)$$

where  $a(t)$ ,  $b(t)$ ,  $c(t)$ ,  $d(t)$ ,  $f(t)$ , and  $g(t)$  are given real-valued functions of time  $t$  only, can be found by a familiar substitution

$$u = Ae^S = A(t) e^{S(x,y,t)} \quad (2.2)$$

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with

$$A = A(t) = \frac{1}{\sqrt{2\pi\mu(t)}} \quad (2.3)$$

and

$$S = S(x, y, t) = \alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2 + \delta(t)x + \varepsilon(t)y + \kappa(t), \quad (2.4)$$

where  $\alpha(t)$ ,  $\beta(t)$ ,  $\gamma(t)$ ,  $\delta(t)$ ,  $\varepsilon(t)$ , and  $\kappa(t)$  are differentiable real-valued functions of time  $t$  only. Indeed,

$$\frac{\partial S}{\partial t} = a \left( \frac{\partial S}{\partial x} \right)^2 - bx^2 + fx + (cx - g) \frac{\partial S}{\partial x} \quad (2.5)$$

provided

$$\frac{\mu'}{2\mu} = -a \frac{\partial^2 S}{\partial x^2} - d = -2\alpha(t)a(t) - d(t). \quad (2.6)$$

Equating the coefficients of all admissible powers of  $x^m y^n$  with  $0 \leq m + n \leq 2$ , gives the following system of ordinary differential equations

$$\frac{d\alpha}{dt} + b(t) - 2c(t)\alpha - 4a(t)\alpha^2 = 0, \quad (2.7)$$

$$\frac{d\beta}{dt} - (c(t) + 4a(t)\alpha(t))\beta = 0, \quad (2.8)$$

$$\frac{d\gamma}{dt} - a(t)\beta^2(t) = 0, \quad (2.9)$$

$$\frac{d\delta}{dt} - (c(t) + 4a(t)\alpha(t))\delta = f(t) - 2\alpha(t)g(t), \quad (2.10)$$

$$\frac{d\varepsilon}{dt} + (g(t) - 2a(t)\delta(t))\beta(t) = 0, \quad (2.11)$$

$$\frac{d\kappa}{dt} + g(t)\delta(t) - a(t)\delta^2(t) = 0, \quad (2.12)$$

where the first equation is the familiar Riccati nonlinear differential equation; see, for example, [12], [18], [22], [23], [27] and references therein.

We have

$$4a\alpha' + 4ab - 2c(4a\alpha) - (4a\alpha)^2 = 0, \quad 4a\alpha = -2d - \frac{\mu'}{\mu}$$

from (2.7) and (2.6) and the substitution

$$4a\alpha' = -2d' - \frac{\mu''}{\mu} + \left( \frac{\mu'}{\mu} \right)^2 + \frac{a'}{a} \left( 2d + \frac{\mu'}{\mu} \right)$$

results in the second order linear equation

$$\mu'' - \tau(t)\mu' - 4\sigma(t)\mu = 0 \quad (2.13)$$

with

$$\tau(t) = \frac{a'}{a} + 2c - 4d, \quad \sigma(t) = ab + cd - d^2 + \frac{d}{2} \left( \frac{a'}{a} - \frac{d'}{d} \right). \quad (2.14)$$

As we shall see later, equation (2.13) must be solved subject to the initial data

$$\mu(0) = 0, \quad \mu'(0) = 2a(0) \neq 0 \quad (2.15)$$

in order to satisfy the initial condition for the corresponding Green function; see the asymptotic formula (2.24) below for a motivation. Then, the Riccati equation (2.7) can be solved by the back substitution (2.6).

We shall refer to equation (2.13) as the *characteristic equation* and its solution  $\mu(t)$ , subject to (2.15), as the *characteristic function*. As the special case (2.13) contains the generalized equation of hypergeometric type, whose solutions are studied in detail in [20]; see also [1], [19], [25], and [27].

Thus, the Green function (fundamental solution or heat kernel) is explicitly given in terms of the characteristic function

$$u = K(x, y, t) = \frac{1}{\sqrt{2\pi\mu(t)}} e^{\alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2 + \delta(t)x + \varepsilon(t)y + \kappa(t)}. \quad (2.16)$$

Here

$$\alpha(t) = -\frac{1}{4a(t)} \frac{\mu'(t)}{\mu(t)} - \frac{d(t)}{2a(t)}, \quad (2.17)$$

$$\beta(t) = \frac{1}{\mu(t)} \exp\left(\int_0^t (c(\tau) - 2d(\tau)) d\tau\right), \quad (2.18)$$

$$\begin{aligned} \gamma(t) = & -\frac{a(t)}{\mu(t)\mu'(t)} \exp\left(2\int_0^t (c(\tau) - 2d(\tau)) d\tau\right) \\ & -4\int_0^t \frac{a(\tau)\sigma(\tau)}{(\mu'(\tau))^2} \left(\exp\left(2\int_0^\tau (c(\lambda) - 2d(\lambda)) d\lambda\right)\right) d\tau, \end{aligned} \quad (2.19)$$

$$\begin{aligned} \delta(t) = & \frac{1}{\mu(t)} \exp\left(\int_0^t (c(\tau) - 2d(\tau)) d\tau\right) \int_0^t \exp\left(-\int_0^\tau (c(\lambda) - 2d(\lambda)) d\lambda\right) \\ & \times \left(\left(f(\tau) + \frac{d(\tau)}{a(\tau)}g(\tau)\right)\mu(\tau) + \frac{g(\tau)}{2a(\tau)}\mu'(\tau)\right) d\tau, \end{aligned} \quad (2.20)$$

$$\begin{aligned} \varepsilon(t) = & -\frac{2a(t)}{\mu'(t)}\delta(t) \exp\left(\int_0^t (c(\tau) - 2d(\tau)) d\tau\right) \\ & -8\int_0^t \frac{a(\tau)\sigma(\tau)}{(\mu'(\tau))^2} \exp\left(\int_0^\tau (c(\lambda) - 2d(\lambda)) d\lambda\right) (\mu(\tau)\delta(\tau)) d\tau \\ & +2\int_0^t \frac{a(\tau)}{\mu'(\tau)} \exp\left(\int_0^\tau (c(\lambda) - 2d(\lambda)) d\lambda\right) \left(f(\tau) + \frac{d(\tau)}{a(\tau)}g(\tau)\right) d\tau, \end{aligned} \quad (2.21)$$

$$\begin{aligned} \kappa(t) = & -\frac{a(t)\mu(t)}{\mu'(t)}\delta^2(t) - 4\int_0^t \frac{a(\tau)\sigma(\tau)}{(\mu'(\tau))^2} (\mu(\tau)\delta(\tau))^2 d\tau \\ & +2\int_0^t \frac{a(\tau)}{\mu'(\tau)} (\mu(\tau)\delta(\tau)) \left(f(\tau) + \frac{d(\tau)}{a(\tau)}g(\tau)\right) d\tau \end{aligned} \quad (2.22)$$

with

$$\delta(0) = \frac{g(0)}{2a(0)}, \quad \varepsilon(0) = -\delta(0), \quad \kappa(0) = 0. \quad (2.23)$$

We have used integration by parts in order to resolve the singularities of the initial data; see section 3 for more details. Then the corresponding asymptotic formula is

$$K(x, y, t) = \frac{e^{S(x, y, t)}}{\sqrt{2\pi\mu(t)}} \sim \frac{1}{\sqrt{4\pi a(0)t}} \exp\left(-\frac{(x-y)^2}{4a(0)t}\right) \exp\left(\frac{g(0)}{2a(0)}(x-y)\right) \quad (2.24)$$

as  $t \rightarrow 0^+$ . Notice that the first term on the right hand side is a familiar heat kernel for the diffusion equation with constant coefficients (cf. Eq. (5.2) below).

By the superposition principle, we obtain solution of the Cauchy initial value problem

$$\frac{\partial u}{\partial t} = Qu, \quad u(x, t)|_{t=0} = u_0(x) \quad (2.25)$$

on the infinite interval  $-\infty < x < \infty$  with the general quadratic form  $Q(p, x)$  in (2.1) as follows

$$u(x, t) = \int_{-\infty}^{\infty} K(x, y, t) u_0(y) dy = Hu(x, 0). \quad (2.26)$$

This yields solution explicitly in terms of an integral operator  $H$  acting on the initial data provided that the integral converges and one can interchange differentiation and integration. This integral is essentially the Laplace transform.

In a more general setting, solution of the initial value problem at time  $t_0$

$$\frac{\partial u}{\partial t} = Qu, \quad u(x, t)|_{t=t_0} = u(x, t_0) \quad (2.27)$$

on an infinite interval has the form

$$u(x, t) = \int_{-\infty}^{\infty} K(x, y, t, t_0) u_0(y, t_0) dy = H(t, t_0) u(x, t_0) \quad (2.28)$$

with the heat kernel given by

$$K(x, y, t, t_0) = \frac{1}{\sqrt{2\pi\mu(t, t_0)}} e^{\alpha(t, t_0)x^2 + \beta(t, t_0)xy + \gamma(t, t_0)y^2 + \delta(t, t_0)x + \varepsilon(t, t_0)y + \kappa(t, t_0)}. \quad (2.29)$$

The function  $\mu(t) = \mu(t, t_0)$  is a solution of the characteristic equation (2.13) corresponding to the initial data

$$\mu(t_0, t_0) = 0, \quad \mu'(t_0, t_0) = 2a(t_0) \neq 0. \quad (2.30)$$

If  $\{\mu_1, \mu_2\}$  is a fundamental solution set of equation (2.13), then

$$\mu(t, t_0) = \frac{2a(t_0)}{W(\mu_1, \mu_2)} (\mu_1(t_0)\mu_2(t) - \mu_1(t)\mu_2(t_0)) \quad (2.31)$$

and

$$\mu'(t, t_0) = \frac{2a(t_0)}{W(\mu_1, \mu_2)} (\mu_1(t_0)\mu_2'(t) - \mu_1'(t)\mu_2(t_0)), \quad (2.32)$$

where  $W(\mu_1, \mu_2)$  is the value of the Wronskian at the point  $t_0$ .

Equations (2.17)–(2.22) are valid again but with the new characteristic function  $\mu(t, t_0)$ . The lower limits of integration should be replaced by  $t_0$ . Conditions (2.23) become

$$\delta(t_0, t_0) = -\varepsilon(t_0, t_0) = \frac{g(t_0)}{2a(t_0)}, \quad \kappa(t_0, t_0) = 0 \quad (2.33)$$

and the asymptotic formula (2.24) should be modified as follows

$$\begin{aligned} K(x, y, t, t_0) &= \frac{e^{S(x, y, t, t_0)}}{\sqrt{2\pi\mu(t, t_0)}} \\ &\sim \frac{1}{\sqrt{4\pi a(t_0)(t-t_0)}} \exp\left(-\frac{(x-y)^2}{4a(t_0)(t-t_0)}\right) \exp\left(\frac{g(t_0)}{2a(t_0)}(x-y)\right). \end{aligned} \quad (2.34)$$

We leave the details to the reader.

### 3. DERIVATION OF THE HEAT KERNEL

Here we obtain the above formulas (2.17)–(2.22) for the heat kernel. The first equation is a direct consequence of (2.6) and our equation (2.8) takes the form

$$(\mu\beta)' = (c - 2d)(\mu\beta), \quad (3.1)$$

whose particular solution is (2.18).

From (2.9) and (2.18) one gets

$$\gamma(t) = \int \frac{a(t)}{\mu^2(t)} e^{2h(t)} dt, \quad h(t) = \int_0^t (c(\tau) - 2d(\tau)) d\tau \quad (3.2)$$

and integrating by parts

$$\gamma(t) = - \int \frac{ae^{2h}}{\mu'} d\left(\frac{1}{\mu}\right) = -\frac{ae^{2h}}{\mu\mu'} + \int \left(\frac{ae^{2h}}{\mu'}\right)' \frac{dt}{\mu}. \quad (3.3)$$

But the derivative of the auxiliary function

$$F(t) = \frac{a(t)}{\mu'(t)} e^{2h(t)} \quad (3.4)$$

is

$$F'(t) = \frac{(a' + 2h'a)e^{2h}\mu' - ae^{2h}\mu''}{(\mu')^2} = -\frac{4\sigma a\mu}{(\mu')^2} e^{2h} = -\frac{4\sigma\mu}{\mu'} F \quad (3.5)$$

in view of the characteristic equation (2.13)–(2.14). Substitution into (3.3) results in (2.19).

Equation (2.10) can be rewritten as

$$(\mu e^{-h}\delta)' = \mu e^{-h}(f - 2\alpha g), \quad h = \int_0^t (c - 2d) d\tau \quad (3.6)$$

and its direct integration gives (2.20).

We introduce another auxiliary function

$$G(t) = \mu(t) \delta(t) e^{-h(t)} \quad (3.7)$$

with the derivative given by (3.6). Then equation (2.11) becomes

$$\frac{d\varepsilon}{dt} = -\frac{g}{\mu} e^h + \frac{2a\delta}{\mu} e^h$$

and

$$\varepsilon(t) = - \int \frac{g}{\mu} e^h dt + 2 \int \frac{aG}{\mu^2} e^{2h} dt. \quad (3.8)$$

Integrating the second term by parts one gets

$$\begin{aligned} \int \frac{aG}{\mu^2} e^{2h} dt &= - \int \frac{aG}{\mu'} e^{2h} d\left(\frac{1}{\mu}\right) = - \int FG d\left(\frac{1}{\mu}\right) \\ &= -\frac{FG}{\mu} + \int \frac{(FG)'}{\mu} dt, \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} (FG)' &= F'G + FG' \\ &= -\frac{4a\sigma\mu}{(\mu')^2} (\mu\delta) e^h + \frac{a\mu}{\mu'} e^h f + \frac{d\mu}{\mu'} e^h g + \frac{1}{2} g e^h \end{aligned} \quad (3.10)$$

in view of (3.5) and (3.6). Then substitution (3.10) into (3.9) allows to cancel the divergent integrals. As a result one can resolve the singularity and simplify expression (3.8) to its final form (2.21).

Finally, by (2.12) and (3.7)

$$\kappa(t) = - \int g\delta dt + \int \frac{aG^2}{\mu^2} e^{2h} dt, \quad (3.11)$$

where the last integral can be transformed as follows

$$\int \frac{aG^2}{\mu^2} e^{2h} dt = - \int FG^2 d\left(\frac{1}{\mu}\right) = -\frac{FG^2}{\mu} + \int \frac{(FG^2)'}{\mu} dt \quad (3.12)$$

with

$$\begin{aligned} (FG^2)' &= F'G^2 + 2FGG' = (FG)'G + FGG' \\ &= -\frac{4a\sigma\mu}{(\mu')^2} (\mu\delta)^2 + \frac{2a\mu}{\mu'} (\mu\delta) f + \frac{2d\mu}{\mu'} (\mu\delta) g + \mu g\delta. \end{aligned} \quad (3.13)$$

Substitution (3.12)–(3.13) into (3.11) gives our final expression (2.22).

The details of derivation of the asymptotic formula (2.24) are left to the reader.

#### 4. SPECIAL INITIAL DATA

In the case  $u(x, 0) = u_0 = \text{constant}$ , our solution (2.26) takes the form

$$\begin{aligned} u(x, t) &= \int_{-\infty}^{\infty} K(x, y, t) u_0 dy \\ &= u_0 \frac{e^{\alpha(t)x^2 + \delta(t)x + \kappa(t)}}{\sqrt{2\pi\mu(t)}} \int_{-\infty}^{\infty} e^{(\beta(t)x + \varepsilon(t))y + \gamma(t)y^2} dy \\ &= \frac{u_0}{\sqrt{-2\mu\gamma}} \exp\left(\frac{(4\alpha\gamma - \beta^2)x^2 + 2(2\gamma\delta - \beta\varepsilon)x + 4\gamma\kappa - \varepsilon^2}{4\gamma}\right), \end{aligned} \quad (4.1)$$

provided  $\gamma(t) < 0$ , with the help of an elementary integral

$$\int_{-\infty}^{\infty} e^{-ay^2 + 2by} dy = \sqrt{\frac{\pi}{a}} e^{b^2/a}, \quad a > 0. \quad (4.2)$$

The details of taking the limit  $t \rightarrow 0^+$  in (4.1) are left to the reader.

When  $u(x, 0) = \delta(x - x_0)$ , where  $\delta(x)$  is the Dirac delta function, one gets formally

$$u(x, t) = \int_{-\infty}^{\infty} K(x, y, t) \delta(y - x_0) dy = K(x, x_0, t). \quad (4.3)$$

Thus, in general, the heat kernel (2.16) provides an evolution of this initial data, concentrated originally at a point  $x_0$ , into the entire space for a suitable time interval  $t > 0$ .

## 5. SOME EXAMPLES

Now let us consider several elementary solutions of the characteristic equation (2.13); more complicated cases may include special functions, like Bessel, hypergeometric or elliptic functions [1], [20], [21], and [27]. Among important elementary cases of our general expressions for the Green function (2.16)–(2.22) are the following:

For the traditional diffusion equation

$$\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2}, \quad a = \text{constant} > 0 \quad (5.1)$$

the heat kernel is

$$K(x, y, t) = \frac{1}{\sqrt{4\pi at}} \exp\left(-\frac{(x - y)^2}{4at}\right), \quad t > 0. \quad (5.2)$$

Equation (4.1) gives the steady solution  $u_0 = \text{constant}$  for all times  $t \geq 0$ . See [3] and references therein for a detailed investigation of the classical one-dimensional heat equation.

The diffusion-type equation

$$\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} + f x u, \quad (5.3)$$

where  $a > 0$  and  $f$  are constants (see [7], [8], [9], [10], [11], [6] and references therein regarding to similar cases of the Schrödinger equation), has the characteristic function of the form  $\mu = 2at$ . The heat kernel is

$$K(x, y, t) = \frac{1}{\sqrt{4\pi at}} \exp\left(-\frac{(x - y)^2}{4at}\right) \exp\left(\frac{f}{2}(x + y)t + \frac{af^2}{12}t^3\right) \quad (5.4)$$

provided  $t > 0$ . Evolution of the uniform initial data  $u(x, 0) = u_0 = \text{constant}$  is given by

$$u(x, t) = u_0 e^{fxt + af^2t^3/3}. \quad (5.5)$$

The initial value problem for the following diffusion-type equation with variable coefficients

$$\frac{\partial u}{\partial t} = a \left( \frac{\partial^2 u}{\partial x^2} - x^2 u \right) + \omega \left( \cosh((2a - 1)t) xu + \sinh((2a - 1)t) \frac{\partial u}{\partial x} \right), \quad (5.6)$$

where  $a > 0$  and  $\omega$  are two constants, was solved in [15] by using the eigenfunction expansion method and a connection with the representations of the Heisenberg–Weyl group  $N(3)$ . Here we apply a different approach. The solution of the characteristic equation

$$\mu'' - 4a^2\mu = 0 \quad (5.7)$$

is  $\mu = \sinh(2at)$  and the corresponding heat kernel is given by

$$\begin{aligned} K(x, y, t) = & \frac{1}{\sqrt{2\pi \sinh(2at)}} \exp\left(-\frac{(x^2 + y^2) \cosh(2at) - 2xy}{2 \sinh(2at)}\right) \\ & \times \exp\left(2\omega \frac{x \sinh(t/2) + y \sinh((2a - 1/2)t)}{\sinh(2at)} \sinh\left(\frac{t}{2}\right)\right) \\ & \times \exp\left(-2\omega^2 \frac{\cosh(2at)}{\sinh(2at)} \sinh^4\left(\frac{t}{2}\right)\right) \\ & \times \exp\left(\frac{\omega^2}{2} \left(t - 2 \sinh t + \frac{1}{2} \sinh(2t)\right)\right), \quad t > 0. \end{aligned} \quad (5.8)$$

Indeed, by (2.17)–(2.19)

$$\alpha = \gamma = -\frac{\cosh(2at)}{2 \sinh(2at)}, \quad \beta = \frac{1}{\sinh(2at)}. \quad (5.9)$$

In this case

$$\begin{aligned} f\mu + \frac{g}{2a}\mu' &= \omega (\cosh((2a - 1)t) \sinh(2at) - \sinh((2a - 1)t) \cosh(2at)) \\ &= \omega \sinh t \end{aligned}$$

and equation (2.20) gives

$$\delta = \omega \frac{\cosh t - 1}{\sinh(2at)} = 2\omega \frac{\sinh^2(t/2)}{\sinh(2at)}. \quad (5.10)$$

By (2.21)

$$\begin{aligned} \varepsilon &= \omega \frac{1 - \cosh t}{\sinh(2at) \cosh(2at)} \\ &\quad + 2a\omega \int_0^t \frac{1 - \cosh \tau}{\cosh^2(2a\tau)} d\tau + \omega \int_0^t \frac{\cosh((2a - 1)\tau)}{\cosh(2a\tau)} d\tau, \end{aligned} \quad (5.11)$$

where the integration by parts gives

$$2a \int_0^t \frac{1 - \cosh \tau}{\cosh^2(2a\tau)} d\tau = (1 - \cosh t) \frac{\sinh(2at)}{\cosh(2at)} + \int_0^t \frac{\sinh(2a\tau)}{\cosh(2a\tau)} \sinh \tau d\tau.$$

Thus

$$\varepsilon = \omega (1 - \cosh t) \frac{\cosh(2at)}{\sinh(2at)} + \omega \int_0^t \frac{\sinh(2a\tau) \sinh \tau + \cosh((2a - 1)\tau)}{\cosh(2a\tau)} d\tau$$

and an elementary identity

$$\sinh(2at) \sinh t + \cosh((2a - 1)t) = \cosh(2at) \cosh t \quad (5.12)$$

leads to an integral evaluation. Two other identities

$$\cosh(2at) \cosh t - \sinh(2at) \sinh t = \cosh((2a - 1)t), \quad (5.13)$$

$$\cosh(2at) - \cosh((2a - 1)t) = 2 \sinh(t/2) \sinh((2a - 1/2)t) \quad (5.14)$$

result in

$$\varepsilon = \omega \frac{\cosh(2at) - \cosh((2a - 1)t)}{\sinh(2at)} = 2\omega \frac{\sinh(t/2) \sinh((2a - 1/2)t)}{\sinh(2at)}. \quad (5.15)$$



In a similar fashion,

$$\kappa = -2\omega^2 \sinh^4(t/2) \frac{\cosh(2at)}{\sinh(2at)} + \frac{1}{2}\omega^2 \left( t - 2 \sinh t + \frac{1}{2} \sinh(2t) \right), \quad (5.16)$$

and equation (5.8) is derived. In the limit  $\omega \rightarrow 0$  this kernel gives also a familiar expression in statistical mechanics for the density matrix for a system consisting of a simple harmonic oscillator [11].

The case  $a = 1/2$  corresponds to the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \left( \frac{\partial^2 u}{\partial x^2} - x^2 u \right) + \omega x u \quad (5.17)$$

and the heat kernel (5.8) is simplified to the form

$$K(x, y, t) = \frac{e^{\omega^2 t/2}}{\sqrt{2\pi \sinh t}} \exp \left( -\frac{((x-\omega)^2 + (y-\omega)^2) \cosh t - 2(x-\omega)(y-\omega)}{2 \sinh t} \right), \quad (5.18)$$

when  $t > 0$ . A similar diffusion-type equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \left( \frac{\partial^2 u}{\partial x^2} + x^2 u \right) + \omega x u \quad (5.19)$$

can be solved with the aid of the kernel

$$K(x, y, t) = \frac{e^{-\omega^2 t/2}}{\sqrt{2\pi \sin t}} \exp \left( -\frac{((x+\omega)^2 + (y+\omega)^2) \cos t - 2(x+\omega)(y+\omega)}{2 \sin t} \right) \quad (5.20)$$

provided  $0 < t < \pi/2$ . We leave the details to the reader.

Following to the case of exactly solvable time-dependent Schrödinger equation found in [17], we consider the diffusion-type equation of the form

$$\frac{\partial u}{\partial t} = \cosh^2 t \frac{\partial^2 u}{\partial x^2} + \sinh^2 t x^2 u + \frac{1}{2} \sinh 2t \left( 2x \frac{\partial u}{\partial x} + u \right). \quad (5.21)$$

The corresponding characteristic equation

$$\mu'' - 2 \tanh t \mu' + 2\mu = 0 \quad (5.22)$$

has two linearly independent solutions

$$\mu_1 = \cos t \sinh t + \sin t \cosh t, \quad (5.23)$$

$$\mu_2 = \sin t \sinh t - \cos t \cosh t \quad (5.24)$$

with the Wronskian  $W(\mu_1, \mu_2) = 2 \cosh^2 t$ , and the first one satisfies the initial conditions (2.15). The heat kernel is

$$K(x, y, t) = \frac{1}{\sqrt{2\pi (\cos t \sinh t + \sin t \cosh t)}} \times \exp \left( \frac{(y^2 - x^2) \sin t \sinh t + 2xy - (x^2 + y^2) \cos t \cosh t}{2 (\cos t \sinh t + \sin t \cosh t)} \right) \quad (5.25)$$

provided  $0 < t < T_1 \approx 0.9375520344$ , where  $T_1$  is the first positive root of the transcendental equation  $\tanh t = \cot t$ . Then  $\gamma(t) < 0$  and the integral (2.26) converges for suitable initial data.

A similar diffusion-type equation

$$\frac{\partial u}{\partial t} = \cos^2 t \frac{\partial^2 u}{\partial x^2} + \sin^2 t x^2 u - \frac{1}{2} \sin 2t \left( 2x \frac{\partial u}{\partial x} + u \right) \quad (5.26)$$

has the characteristic equation of the form

$$\mu'' + 2 \tan t \mu' - 2\mu = 0 \quad (5.27)$$

with the same solution (5.23). It appeared in [17] and [6] for a special case of the Schrödinger equation. The corresponding heat kernel has the same form (5.25) but with  $x$  and  $y$  interchanged:

$$\begin{aligned} K(x, y, t) = & \frac{1}{\sqrt{2\pi (\cos t \sinh t + \sin t \cosh t)}} \\ & \times \exp \left( \frac{(x^2 - y^2) \sin t \sinh t + 2xy - (x^2 + y^2) \cos t \cosh t}{2 (\cos t \sinh t + \sin t \cosh t)} \right) \end{aligned} \quad (5.28)$$

provided  $0 < t < T_2 \approx 2.347045566$ , where  $T_2$  is the first positive root of the transcendental equation  $\tanh t = -\cot t$ . We leave the details for the reader.

## 6. SOLUTION OF THE NON-HOMOGENEOUS EQUATION

A diffusion-type equation of the form

$$\left( \frac{\partial}{\partial t} - Q(t) \right) u = F, \quad (6.1)$$

where  $Q$  stands for the second order linear differential operator in the right hand side of equation (2.1) and  $F = F(t, x, u)$ , can be rewritten formally as an integral equation (the Duhamel principle; see [4], [5], [13], [14], [24], [26] and references therein)

$$u(x, t) = H(t, 0) u(x, 0) + \int_0^t H(t, s) F(s, x, u) ds. \quad (6.2)$$

Operator  $H(t, s)$  is given by (2.28). When  $F$  does not depend on  $u$ , one gets a solution of the nonhomogeneous equation (6.1).

Indeed, a formal differentiation gives

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} H(t, 0) u(x, 0) + \frac{\partial}{\partial t} \int_0^t H(t, s) F(s, x, u) ds, \quad (6.3)$$

where

$$\frac{\partial}{\partial t} \int_0^t H(t, s) F(s, x, u) ds = H(t, t) F(t, x, u) + \int_0^t \frac{\partial}{\partial t} H(t, s) F(s, x, u) ds \quad (6.4)$$

and we assume that  $H(t, t)$  is the identity operator. Also

$$Q(t) u = Q(t) H(t, 0) u(x, 0) + \int_0^t Q(t) H(t, s) F(s, x, u) ds \quad (6.5)$$

and

$$\left( \frac{\partial}{\partial t} - Q(t) \right) u = \left( \frac{\partial}{\partial t} - Q(t) \right) H(t, 0) u(x, 0) + F \quad (6.6)$$

$$+ \int_0^t \left( \frac{\partial}{\partial t} - Q(t) \right) H(t, s) F(s, x, u) \, ds,$$

where

$$\left( \frac{\partial}{\partial t} - Q(t) \right) H(t, s) = 0, \quad 0 \leq s < t \quad (6.7)$$

by construction of the operator  $H(t, s)$  in (2.28). This completes our formal proof. A rigorous proof will be given elsewhere.

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## REFERENCES

- [1] G. E. Andrews, R. A. Askey, and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.
- [2] L. M. A. Bettencourt, A. Cintrón-Arias, D. I. Kaiser, and C. Castillo-Chávez, *The power of a good idea: Quantitative modeling of the spread of ideas from epidemiological models*, *Physica A* **364** (2006), 513–536.
- [3] J. R. Cannon, *The One-Dimensional Heat Equation*, Encyclopedia of Mathematics and Its Applications, Vol. 32, Addison–Wesley Publishing Company, Reading etc, 1984.
- [4] T. Cazenave, *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics, Vol. 10, American Mathematical Society, Providence, Rhode Island, 2003.
- [5] T. Cazenave and A. Haraux, *An Introduction to Semilinear Evolution Equations*, Oxford Lecture Series in Mathematics and Its Applications, Vol. 13, Oxford Science Publications, Clarendon Press, Oxford, 1998.
- [6] R. Cordero-Soto, R. M. Lopez, E. Suazo, and S. K. Suslov, *Propagator of a charged particle with a spin in uniform magnetic and perpendicular electric fields*, *Lett. Math. Phys.* **84** (2008) #2–3, 159–178.
- [7] R. P. Feynman, *The Principle of Least Action in Quantum Mechanics*, Ph. D. thesis, Princeton University, 1942; reprinted in: “Feynman’s Thesis – A New Approach to Quantum Theory”, (L. M. Brown, Editor), World Scientific Publishers, Singapore, 2005, pp. 1–69.
- [8] R. P. Feynman, *Space-time approach to non-relativistic quantum mechanics*, *Rev. Mod. Phys.* **20** (1948) # 2, 367–387; reprinted in: “Feynman’s Thesis – A New Approach to Quantum Theory”, (L. M. Brown, Editor), World Scientific Publishers, Singapore, 2005, pp. 71–112.
- [9] R. P. Feynman, *The theory of positrons*, *Phys. Rev.* **76** (1949) # 6, 749–759.
- [10] R. P. Feynman, *Space-time approach to quantum electrodynamics*, *Phys. Rev.* **76** (1949) # 6, 769–789.
- [11] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw–Hill, New York, 1965.
- [12] D. R. Haaheim and F. M. Stein, *Methods of solution of the Riccati differential equation*, *Mathematics Magazine* **42** (1969) #2, 233–240.
- [13] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural’ceva, *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, Rhode Island, 1968. (pp. 318, 356)
- [14] E. E. Levi, *Sulle equazioni lineari totalmente ellittiche alle derivate parziali*, *Rend. Circ. Mat. Palermo* **24** (1907) , 275–317.
- [15] R. M. Lopez and S. K. Suslov, *The Cauchy problem for a forced harmonic oscillator*, arXiv:0707.1902v8 [math-ph] 27 Dec 2007.
- [16] I. V. Melnikova and A. Filinkov, *Abstract Cauchy problems: Three Approaches*, Chapman&Hall/CRC, Boca Raton, London, New York, Washington, D. C., 2001.

- [17] M. Meiler, R. Cordero-Soto, and S. K. Suslov, *Solution of the Cauchy problem for a time-dependent Schrödinger equation*, J. Math. Phys. **49** (2008) #7, published on line 9 July 2008, URL: <http://link.aip.org/link/?JMP/49/072102>; see also arXiv: 0711.0559v4 [math-ph] 5 Dec 2007.
- [18] A. M. Molchanov, *The Riccati equation  $y' = x + y^2$  for the Airy function*, [in Russian], Dokl. Akad. Nauk **383** (2002) #2, 175–178.
- [19] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*, Springer-Verlag, Berlin, New York, 1991.
- [20] A. F. Nikiforov and V. B. Uvarov, *Special Functions of Mathematical Physics*, Birkhäuser, Basel, Boston, 1988.
- [21] E. D. Rainville, *Special Functions*, The Macmillan Company, New York, 1960.
- [22] E. D. Rainville, *Intermediate Differential Equations*, Wiley, New York, 1964.
- [23] S. S. Rajah and S. D. Maharaj, *A Riccati equation in radiative stellar collapse*, J. Math. Phys. **49** (2008) #1, published on line 23 January 2008.
- [24] E. Suazo, and S. K. Suslov, *An integral form of the nonlinear Schrödinger equation with variable coefficients*, arXiv:0805.0633v2 [math-ph] 19 May 2008.
- [25] S. K. Suslov and B. Trey, *The Hahn polynomials in the nonrelativistic and relativistic Coulomb problems*, J. Math. Phys. **49** (2008) #1, published on line 22 January 2008, URL: <http://link.aip.org/link/?JMP/49/012104>.
- [26] T. Tao, *Nonlinear Dispersive Equations: Local and Global Analysis*, CBMS regional conference series in mathematics, 2006.
- [27] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Second Edition, Cambridge University Press, Cambridge, 1944.

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